Anisomagnetic quasi-achromats with small effective emittance

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1. Introduction

The horizontal emittance, ϵ_x, is an important parameter of any synchrotron light source. Its value depends on the magnetic structure (lattice) of a storage ring, and first of all – on parameters and arrangement of dipole magnets. For achromatic lattices DBA (double bend achromat – two dipole magnets per cell), TBA, QBA, where the 'intercell' regions have zero dispersion, the conditions of natural emittance minimum are well known from the works of Teng, and Lee, [1, 2]. The emittance strongly decreases with increase of the number of dipole magnets in a ring, $\epsilon_x \propto N^{-3}$.

The figure of merit of an insertion device (ID) is defined, however, by the *effective emittance*, which depends on dispersion (or Courant-Snider invariant) in the ID straight section. It turns out that the effective emittance can be lowered if to weaken the dispersion-free condition [3]. Therefore, the beam optics of some electron rings, initially achromatic, of DBA kind, was modified (so called distributed dispersion lattice, or quasi-achromat, DBA*).

In this work the problem of minimization of the effective emittance is to be solved for the case when the lattice cell is symmetric (with respect to inversion in its center) and includes magnets of two types: *internal* dipoles M_0 and a pair of the *end* (or side) dipoles M_1; the total number of magnets in a cell is $m = m_0 + 2$. Modified lattices TBA* and DBA* correspond to $m_0 = 1$ and $m_0 = 0$, respectively. In the case $m_0 = 2$, the internal dipoles can also have non-zero shift; this case will be denoted as QBA**.

Dipoles with unequal fields serve as SR sources with different critical energy, i.e, with peak regions in different parts of X-ray spectrum, so they can serve better for different 'classes' of SR users. However, large number of free parameters, which describe anisomagnetic cell, makes the search of the best parameters (when the effective emittance is minimal) a quite difficult problem. Here a possible way, an algorithm to solve this problem is suggested, which divides the problem into a few more simple parts.

2. Effective emittance

The horizontal emittance, ϵ_x, and the effective (horizontal) emittance, $\epsilon_{\rm eff}$, are defined by the next expressions, relating to some integrals along the orbit (they depend on the orbit's radius, ρ, and optics function, including β-function) [2, 3]:

$$\epsilon_x = C_q\,\gamma^2\,\varepsilon_x,\;\; \varepsilon_x = \frac{I_5}{J_x I_2} = \frac{I_5}{I_2 - I_4}, \tag{1}$$

$$\epsilon_{\rm eff} = C_q\,\gamma^2\,\varepsilon_{\rm eff}\,, \quad \varepsilon^2_{\rm eff} = \varepsilon_x\left(\varepsilon_x + \frac{H_{\rm ID}\,I_3}{2I_2 + I_4}\right). \tag{2}$$

Here γ is relativistic factor, $C_q = 3.84\times 10^{-13}\,$m, $C_q\gamma^2 = 1470\,$nm$\,(E[{\rm GeV}])^2$; these are the usual notations of some ring integrals:

$$I_2 = \oint \rho^{-2}\,ds,\quad I_4 = \oint \eta\rho^{-3}(1+2k\rho^2)\,ds,\quad I_3 = \oint |\rho|^{-3}\,ds\,,\; I_5 = \oint |\rho|^{-3} H\,ds\,; \tag{3}$$

$H_{\rm ID}$ in (2) is the Courant–Snyder invariant in the region of insertion devise ($\alpha = -\frac{1}{2}\beta'$; η is dispersion function),

$$H = \beta^{-1}\{\eta^2 + (\alpha\eta + \beta\eta')^2\}. \tag{4}$$

Some rings, sources of SR, were designed that the regions of ID would have invariant $H_{\rm ID}$ vanishing (i.e., dispersion and its derivative are both zero), but in the curse of time, lattice parameters were changed to reach better minimization of the effective emittance (2).

Integral I_4 is small in comparison with I_2, so it can be neglected, in the case of separate function magnets (dipoles have no quadrupole component, and $k = 0$). So, we seek to minimize the next combination of ring integral:

$$\varepsilon^2_{\rm eff} = I_5 I_6/I_2^2 = I_5(I_5 + H_{\rm ID}\,I_3/2)/I_2^2\,; \tag{5}$$

and here one should note that integrals I_2 and I_3 depends only on the field of dipoles (radius of the orbit), but does not depend on betatron and dispersion functions. On the other hand, integrals I_5 and $I_6 = I_5 + H_{\rm ID}\,I_3/2$ are a sum of positive contributions from all magnets (because the Courant-Snyder invariant is positive).

Therefore this problem of optimization can be attempted to be divided into a few relatively simple steps, that is, to be factorized:

(1) Firstly, we make minimization of a magnet's contribution to I_5 with respect to parameters β_0, η_0, η_0', i.e., the value of beta-function at its minimum, and the dispersion functions in the point of this minimum (this point serves as the reference point, zero point of the orbit coordinate s in a dipole); for the end magnets, which neighbour ID straight sections and directly define invariant $H_{\rm ID}$, we minimize their contribution to the sum $I_5 + I_6$, because it *approximates* in a good way the necessary minimization of the product $I_5 I_6$ (the arithmetic mean majorizes the geometric mean).

(2) The other dipole's parameters – the curvature ρ^{-1} (or the field), the length or the angle of rotation, $l = \vartheta\rho$, as well as the the coordinate of dipole's center, s_0, or the (dimensionless parameter) *shift*, $x = -2s_0/l$, – should be restricted with the matching condition, which states the equality of the Courant–Snyder invariants on the exit from a dipole and on the entrance to the next one. One more restriction is that the orbit is closed, i.e., the sum of angles ϑ_i of all dipole magnets is equal to 2π.

(3) After eliminating the angles, one can minimize the effective emittance with respect to the rest (two or three) parameters: $\kappa = \rho_1/\rho_0$, $x = x_0$, $y = x_1$ (the inner magnets can be shifted, x_0, if their number is two).

It is known that the contribution of a dipole magnet to the (natural) emittance is minimal when its center coincide with the minimum of beta-function (besides, this minimum and the dispersion parameters should have definite values depending on the magnet's field [1, 2]). Such a magnet can be called *symmetric*, or *unshifted*; otherwise, we have a magnet with a shift.

As a rule, lattices consist of repeating, equal collections of dipole magnets – cells or periods (or superperiods). We will consider quite general case of a periodic lattice of n symmetric cells (inversion in the cell center). Every cell contains m dipole magnets, including $m_0 = m - 2$ internal magnets M_0 (unshifted, as a rule), and two (one on each side) end magnets M_1, with a shift x_1. Considering the right half of the cell, we will imply that the shift of the end magnet is positive (and it is negative in the left half of the cell). The total number of dipoles is $N = n\cdot m$; their rotation angles, ϑ_0, ϑ_1, should comply with the requirement:

$$\sum \vartheta_i = n(m_0\vartheta_0 + 2\vartheta_1) = 2\pi\,. \tag{6}$$

We will use the notation of the *mean* angle: $\bar\vartheta = 2\pi/N$.

3. Minimization of contribution (to the emittance) of the internal dipoles

Let the length of a magnet is $l = \rho\vartheta$, and the coordinate of its center (from the minimum of beta-function) is $s_0 = -xl/2$. Following Teng [1], one can write the contribution of a magnet to integral I_5, as well as the Courant–Snyder invariant at its right and left edges, $H_\pm$, in the next form (see the expression (9)–(11) from [1]):

$$H_\pm = \beta_0(\eta_0' + \sin\varphi_\pm)^2 + \rho^2\beta_0^{-1}\left(1 - \eta_0\rho^{-1} - \cos\varphi_\pm\right)^2, \tag{7}$$

$$\begin{aligned} \Delta I_5 \;&\equiv\; \frac{1}{\rho^3}\int_{s_-}^{s_+} H\,ds = \frac{\vartheta\beta_0}{\rho^2}[A + D + (\eta_0' + E\sin\varphi_0)^2] \\ &+\; \frac{\vartheta}{\beta_0}[A - D + (1 - \frac{\eta_0}{\rho} - E\cos\varphi_0)^2],\;\text{where } s_\pm = s_0 \pm \frac{l}{2},\;\varphi_\pm = \frac{s_\pm}{\rho},\;\varphi_0 = \frac{s_0}{\rho} = -\frac{x\vartheta}{2}, \end{aligned} \tag{8}$$

$$\begin{cases} A = \frac{1}{2} - \frac{1-\cos\vartheta}{\vartheta^2} \approx \frac{\vartheta^2}{4!} - \frac{\vartheta^4}{6!},\quad B = \frac{1}{2} - A - \frac{\sin\vartheta}{2\vartheta} \approx \frac{\vartheta^2}{4!} - \frac{2\vartheta^4}{6!}, \\ D = B\cos 2\varphi_0 \approx \frac{\vartheta^2}{24}\left[1 - \frac{\vartheta^2}{30}(2 + 15x^2)\right],\quad E = \frac{\sin(\vartheta/2)}{\vartheta/2} \approx 1 - \frac{\vartheta^2}{24} + \frac{\vartheta^4}{16\cdot 5!}. \end{cases} \tag{9}$$

It is assumed that angles ϑ_i are small enough (for all dipoles of the storage ring), so already the first terms of angle expansion gives a sufficient accuracy.

Minimization of expression (8) with respect to η_0, η_0' is reached if these parameters complies with the next conditions [1]:

$$\eta_0' = -E\sin\varphi_0 \approx \frac{x\vartheta}{2},\quad \eta_0 = \rho - \rho E\cos\varphi_0 \approx \rho\vartheta^2\frac{1 + 3x^2}{24}\,. \tag{10}$$

The dipole's contribution to the integral takes the form $\Delta I_5 = \frac{\vartheta}{\rho}\left\{\frac{\beta_0}{\rho}(A + D) + \frac{\rho}{\beta_0}(A - D)\right\}$; further minimization with respect to β_0 gives now

$$\beta_0^2 \;=\; \rho^2\frac{A - D}{A + D} \approx \rho^2\vartheta^2\frac{1 + 15x^2}{60}, \tag{11}$$

$$\Delta I_5^{(\rm min)} \;=\; \frac{2\vartheta}{\rho}\sqrt{A^2 - D^2} \approx \frac{\vartheta^4\sqrt{1 + 15x^2}}{12\sqrt{15}\rho}. \tag{12}$$

Taking into account (10), (11), one can reduce equation (7) to the following one:

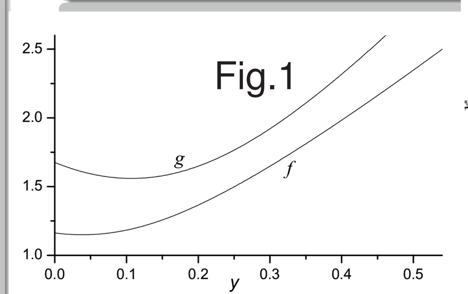
$$H_\pm \approx \rho\vartheta^3\frac{4 \mp 15x + 45x^2}{12\sqrt{15}\sqrt{1 + 15x^2}}\,. \tag{13}$$

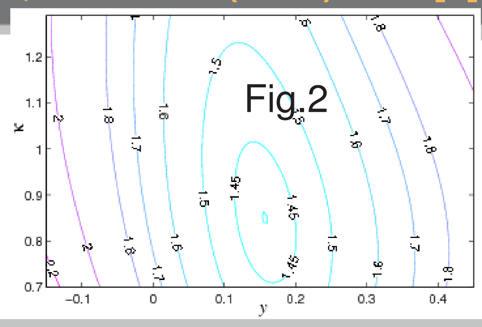
At $x = 0$, we obtain the expressions for the central (unshifted) magnet (the case A of Teng). A lattice composed of N such magnets (let $\rho = 1$, then $I_2 = I_3 = N\vartheta$, $I_5 = N\Delta I_5$), corresponds to the case of minimal natural emittance,

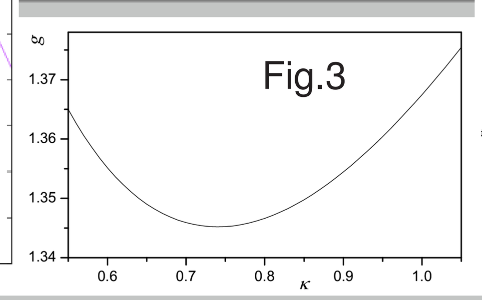
$$\varepsilon_0 = \frac{\bar\vartheta^{\,3}}{12\sqrt{15}}\left(= \frac{I_5}{I_2}\right);\quad \bar\vartheta = \frac{2\pi}{N}\,(= \vartheta); \tag{14}$$

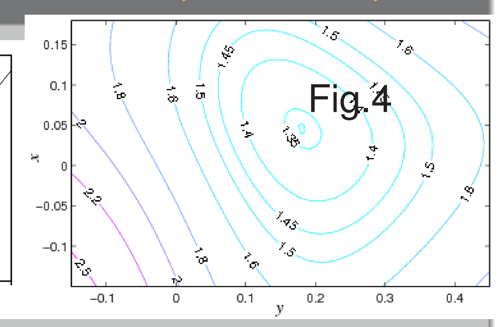
the effective emittance of this lattice is equal to $3\varepsilon_0$.

[1] L. Teng, Argonne Lab. Report LS-17 (1985). [2] S.Y. Lee, Phys. Rev. E54, 1940 (1996). [3] H. Tanaka and A. Ando, NIM A369 (1996) 312. [4] Y. Papaphilippou and P. Elleaume, PAC 2005, 2086









It is convenient to measure emittances in units of ε_0, through introduction of dimensionless values f, g. It is known [2], that for the DBA lattice

$$f \equiv \frac{\varepsilon_x}{\varepsilon_0} = 3,\quad g \equiv \frac{\varepsilon_{\rm eff}}{\varepsilon_0} = 3.$$

Dipoles of DBA-lattice have the shift $x = \frac{1}{4}$; using this value $x = \frac{1}{4}$ in (10)–(13), one can obtain (the effective emittance of such a quasi-achromat lattice is smaller in comparison with DBA case) $f = \frac{\sqrt{31}}{4}$, $g = \frac{111}{8\sqrt{31}} \approx 2.49$. At the point of minimum of $H_+(x)$, see (13), $x = 0.229$ (this is the root of equation $45x^3 + 2x = 1$), the effective emittance becomes yet some smaller: $g = 2.43$. However, the more exact approach to minimize DBA* is to use another way to choose the optics parameters – as for the end dipoles.

4. Minimization of the end dipoles' contribution to the effective emittance

Let the integral over all cell dipoles but the end ones is equal to I_5^*, and the end dipole's contribution is ΔI_5. Accounting for equation (5), we have

$$\Delta(I_5 I_6) = n^2(I_5^* + 2\Delta I_5)(I_5^* + 2\Delta I_6) - n^2(I_5^*)^2 \approx 2n^2 I_5^*(\Delta I_5 + \Delta I_6) \propto 2\Delta I_5 + JH_+/2\,.$$

Here $J = I_3/(2n)$ – integral along the half-cell, n is the number of cells. Using (7)–(9), one can find the expression, which should be minimized (with respect to dipole's parameters η_0, η_0', β_0; for simplicity sake, take for a while $\rho = 1$):

$$\begin{aligned} \Delta I_5 \;&+\; \Delta I_6 = 2\vartheta\beta_0[A + D + (\eta_0' + E\sin\varphi_0)^2] + \frac{2\vartheta}{\beta_0}[A - D + (1 - \eta_0 - E\cos\varphi_0)^2] \\ &+\; \frac{J\beta_0}{2}(\eta_0' + \sin\varphi_+)^2 + \frac{J}{2\beta_0}(1 - \eta_0 - \cos\varphi_+)^2. \end{aligned} \tag{15}$$

Zeroing the derivatives by η_0' and η_0, one can find the values of these parameters when the minimum occurs (restore ρ):

$$\eta_0' \approx \frac{\vartheta}{2}\cdot\frac{4y\vartheta - J\rho^2(1 - y)}{4\vartheta + J\rho^2}\,,\quad \eta_0 \approx \frac{\rho\vartheta^2}{24}\cdot\frac{4\vartheta(1 + 3y^2) + 3J\rho^2(1 - y)^2}{4\vartheta + J\rho^2}\,; \tag{16}$$

here y is the shift of end dipoles. Substituting this values to (15) one can obtain:

$$\Delta I_5 + \Delta I_6 = \frac{\vartheta^3}{6}\left\{4\beta_0\frac{\vartheta + J}{4\vartheta + J} + \frac{\vartheta^2}{60\beta_0}\left[1 + 15y^2 + \frac{5J(1 - 3y)^2}{4\vartheta + J}\right]\right\}\,; \tag{17}$$

the minimum is reached if (restore ρ)

$$\beta_0^2 = \rho^2\vartheta^2\frac{2\vartheta(1{+}15y^2) + 3J\rho^2(1{-}5y{+}10y^2)}{120(\vartheta + J\rho^2)}\,. \tag{18}$$

The edge CS-invariant takes the form:

$$H_\pm = \frac{\beta_0\vartheta^2}{4}\left[\frac{4\vartheta + J \mp J}{4\vartheta + J}\right]^2 + \frac{\vartheta^4}{9\beta_0}\left[\frac{\vartheta \mp 3y\vartheta + (3 \mp 3)yJ/4}{4\vartheta + J}\right]^2. \tag{19}$$

In principle, now all is ready for the final minimization: in addition to equations (17)–(19) one has to take into account that $\Delta I_6 - \Delta I_5 = JH_+/2\,$.

5. Minimization for DBA* lattice; Lattices TBA*, QBA*, et cet. QBA** lattice

In order to check the appropriateness of the equations of the previous section, let us consider the simplest quasi-achromat lattice, with two magnets in a cell. In this case (again $\rho = 1$) $J \equiv I_3/N = \vartheta = \bar\vartheta$, and equations (16)–(19) lead to

$$\eta_0' = \vartheta\frac{5y - 1}{10}\,,\quad \eta_0 = \vartheta^2\frac{7 - 6y + 15y^2}{120}\,,\quad \beta_0 = \frac{\vartheta\sqrt{1 - 3y + 12y^2}}{4\sqrt{3}}\,.$$

$$\Delta I_5 + \Delta I_6 = 2\vartheta^4\frac{\sqrt{1 - 3y + 12y^2}}{15\sqrt{3}},\quad H_+ = \vartheta^3\frac{7 - 33y + 72y^2}{75\sqrt{3}\sqrt{1 - 3y + 12y^2}}.$$

As a result, we find the expressions for the 'simple' dimensionless emittances (natural and effective, respectively), see (14):

$$f = \frac{\Delta I_5}{\vartheta\varepsilon_0} = \frac{13 - 27y + 168y^2}{5\sqrt{5{-}15y{+}60y^2}}\,,\; g^2{=}\frac{\Delta I_5\Delta I_6}{\vartheta^2\varepsilon_0^2} = \frac{3(13 - 27y + 168y^2)(9 - 31y + 104y^2)}{125(1 - 3y + 12y^2)}\,. \tag{20}$$

The corresponding curves are shown at the fig. 1. The minimum of effective emittance if reached at $y = 0.107$, and its value is $g_{\rm min} = 1.559$. So, the suggested minimization algorithm reproduces with an accuracy better than half a percent the minimum of Tanaka–Ando [4], which is 1.552.

Further we will consider lattices where the cell includes internal dipoles, with zero shift. Let m_0 is the number of internal dipoles, and $m = m_0 + 2$ is the total number of dipoles in a cell. We introduce the next dimensionless parameters: $q = \vartheta_1/\bar\vartheta$, $p = \vartheta_0/\vartheta_1$, $\kappa = \rho_1/\rho_0$, and let $\rho_1 = 1$ (unit length). We will also omit in equations the mean angle (i.e., taking formally $\bar\vartheta = 1$), because it should not enter into the simple emittance, both f and g. Still it is impossible to find $g(y,\kappa;\,m_0)$ in an explicit form, but one can find this function through a numerical computing, according the following algorithm.

The orbit's closure condition and the first integrals along the half-cell, normalized on ϑ_1, see (3), read:

$$q = \frac{m}{m_0 p + 2}\,,\quad i_2 \equiv I_2/(2n\vartheta_1) = 1 + \frac{m_0}{2}p\kappa\,,\quad j \equiv J/\vartheta_1 = I_3/(2n\vartheta_1) = 1 + \frac{m_0}{2}p\kappa^2\,. \tag{21}$$

Equation (18) gives (after normalization)

$$b \equiv \frac{\beta_0^{(1)}}{\vartheta_1} = \sqrt{\frac{2 + 30y^2 + 3j(1 - 5y + 10y^2)}{120(1 + j)}}\,.$$

In the same way (19) and (17) transform to

$$h_\pm \equiv \frac{H_\pm^{(1)}}{\vartheta_1^3},\quad S \equiv \frac{\Delta I_5^{(1)} + \Delta I_6^{(1)}}{\vartheta_1^4} = \frac{4b(1 + j)}{3(4 + j)}\,.$$

The matching condition $H_+^{(0)} = H_-^{(1)}$ gives

$$\frac{p^3}{3\sqrt{15}\kappa} = b\frac{(2 + j)^2}{(4 + j)^2} + \frac{(2 + 6y + 3jy)^2}{36b(4 + j)^2}\;(= h_-). \tag{22}$$

Equation (22) subject to (21) can be solved numerically; this gives a unique solution $p(y,\kappa;\,m_0)$. Now, one can find $b(y,\kappa;\,m_0)$ (the minimum of beta-function of the end dipole, $\beta_0^{(1)}$), and, at last, calculate f and g:

$$i_5 = \frac{I_5}{2n\vartheta_1} = \frac{q^3}{4}(2m_0\varepsilon_0 p^4\kappa + 2s - jh_+), \tag{23}$$

$$f = \frac{i_5}{\varepsilon_0 i_2}\,,\quad g^2 = f\left(f + \frac{q^3 jh_+}{2i_2\varepsilon_0}\right);\;\varepsilon_0{=}\frac{1}{12\sqrt{15}}\,. \tag{24}$$

Fig. 2 shows the region of the effective emittance minimum, $g(y,\kappa)$, for TBA* lattice. The Table contains parameters, including the natural emittance, for quasi-achromat lattices mBA* up to $m = 10$.

Table 1: Parameters of quasi-achromats mBA*

m	2	3	4	5	6	7	8	9	10
$g_{\rm min}$	1.559	1.435	1.353	1.297	1.257	1.226	1.202	1.182	1.166
f	1.191	1.207	1.196	1.182	1.169	1.156	1.146	1.136	1.128
y	0.107	0.157	0.185	0.203	0.215	0.224	0.231	0.236	0.240
κ	–	0.850	0.835	0.830	0.831	0.832	0.835	0.839	0.843
p	–	1.134	1.170	1.198	1.222	1.241	1.258	1.272	1.285

If $m = 4$, i.e., four dipoles in a cell, the internal dipoles can also have a non-zero shift, $x_0 = x$ (the shift of side dipoles is $x_1 = y$). This time, one should change the left hand side of equation (22) using equation (13):

$$\frac{p^3(4 - 15x + 45x^2)}{12\kappa\sqrt{15}\sqrt{1 + 15x^2}} = h_-(j, y).$$

Moreover, in the expression for i_5, eq-n (23), the contribution of internal dipoles should be changed [see(12)]:

$$i_5 = q^3(2m_0\varepsilon_0 p^4\kappa\sqrt{1 + 15x^2} + 2s - jh_+)/4.$$

Now we need to find a minimum of the function of three variables, $g(y, x, \kappa)$. One can consider two-dimensional sections, for different values of κ, seeking for the minimum $g_{\rm min}(\kappa)$; the plot of this function is shown on the fig. 3. Fig. 4 shows the region of g-minimum for the case $\kappa = 0.74$ (one can compare with the fig. 2). The minimum itself, $g_{\rm min} = 1.345$, is reached at $y = 0.181$, $x = 0.046$ (at that $p = 1.184$).

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